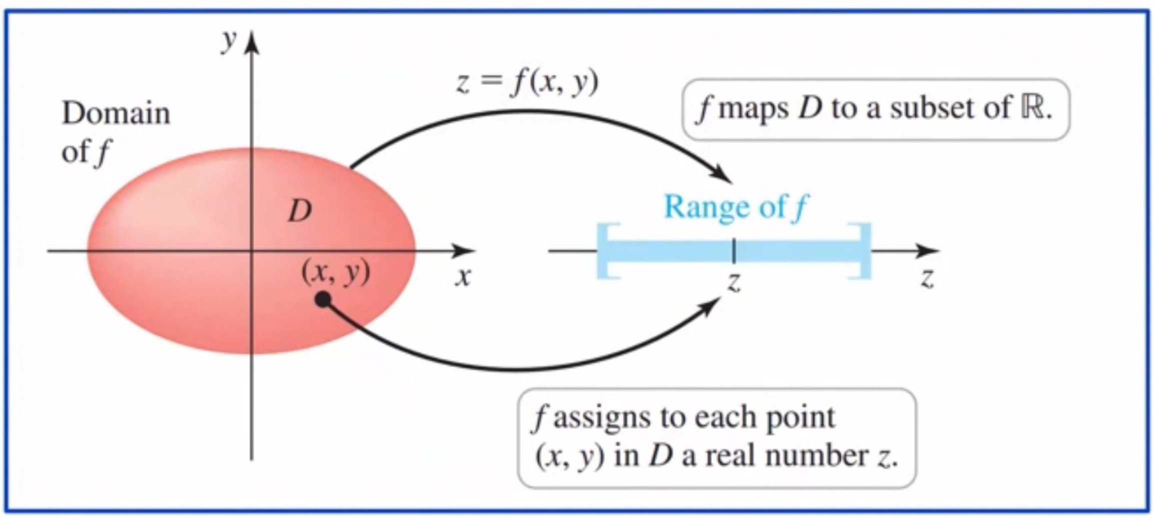
15.1 Graphs and Level Curves

Functions of Two Variables

DEFINITION Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.



E.g.)

Find the domain

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2-25}}.$$

Ex: $f(4.5) = \frac{1}{\sqrt{4^2+5^2-25}}$

$= \frac{1}{\sqrt{16+25-25}}$

$= \frac{1}{\sqrt{16}} = \frac{1}{4} \in \mathbb{R}.$

(4.5) is in the domain.

Ex: $f(3,4) = \frac{1}{\sqrt{3^2+4^2-25}}$

$= \frac{1}{\sqrt{9+16-25}} = \frac{1}{\sqrt{25-25}} = \frac{1}{\sqrt{0}}.$

(3.4) is not in the domain

Domain: $x^2+y^2-25>0.$

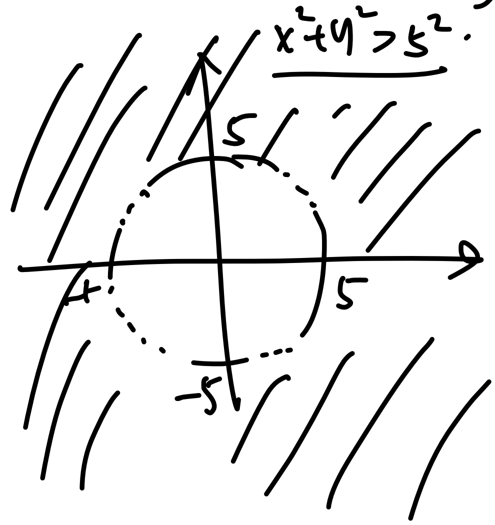
① $\sqrt{\ } \geq 0$

② $\frac{1}{\boxed{\sqrt{\ }}} \neq 0$

$x^2+y^2-25>0.$

$D = \{(x,y) : x^2+y^2-25>0\}.$

$x^2+y^2>5^2.$



e.g.)

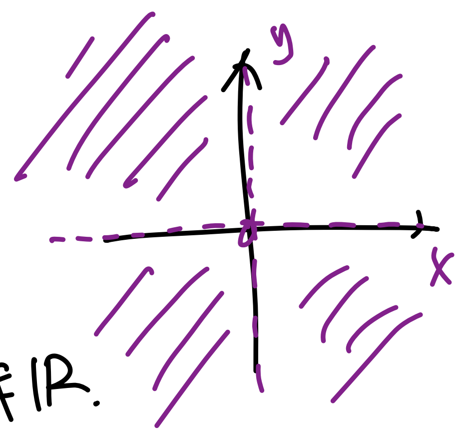
$$g(x,y) = \frac{1}{xy}.$$

$$g(1.1) = \frac{1}{1\cdot 1} = 1 \in \mathbb{R}$$

$(1.1) \in$ Domain.

$$g(1.0) = \frac{1}{1\cdot 0} = \frac{1}{0} \notin \mathbb{R}.$$

$(1.0) \notin$ Domain.



<u>Domain</u>: we can't have 0 on the denom.

so, $xy \neq 0 \implies x \neq 0$ <u>and</u> $y \neq 0$

$$D = \{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}.$$

<u>Range</u>:

$\frac{1}{xy}$ can be anything but can't be zero.

$$R = \mathbb{R} \setminus \{0\}.$$

$$= \{z : z \neq 0\}$$

e.g.)

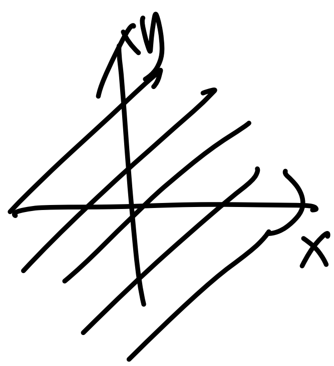
$z = 4x^2 + 4y^2$

$= f(x,y)$

Domain: x and y can be anything

z will always be defined.

so, $\mathbb{R}^2$ (meaning any x, y in $\mathbb{R}$).



Range:

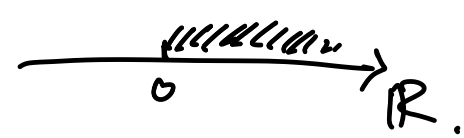
$x^2 \geq 0$

$y^2 \geq 0$

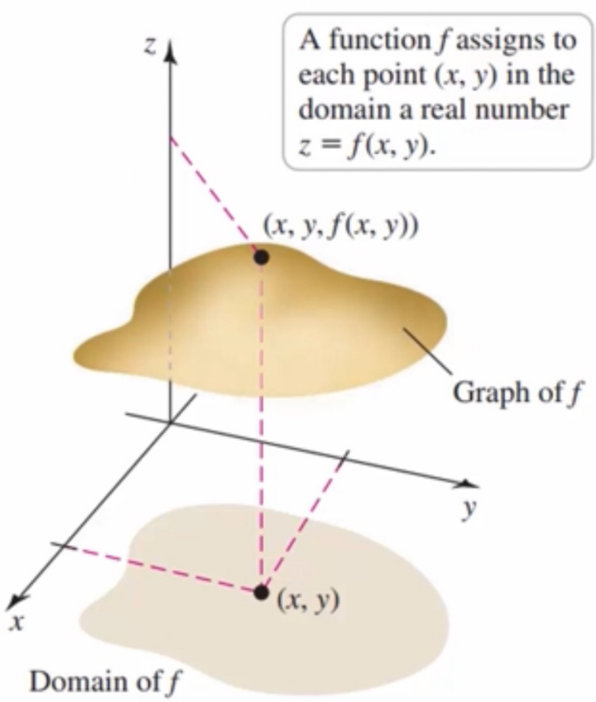
$4x^2 + 4y^2 \geq 0$.

meaning 0 or greater.

$R = \{z : z \geq 0\}$

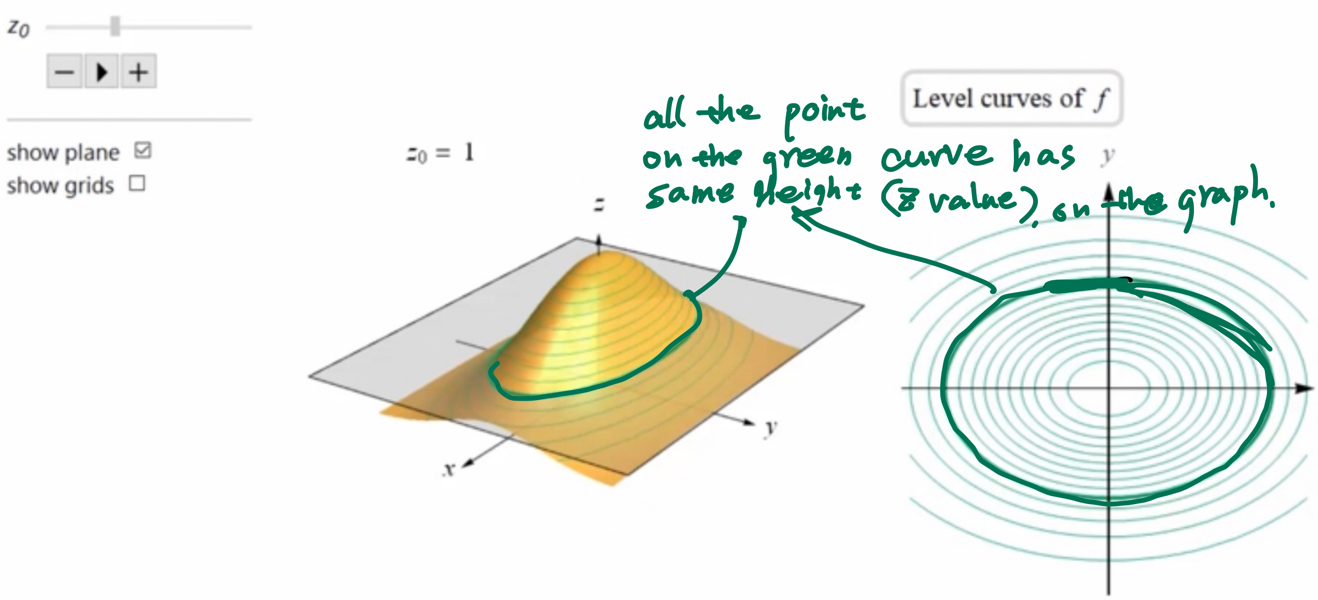


Graphs of functions of two variables

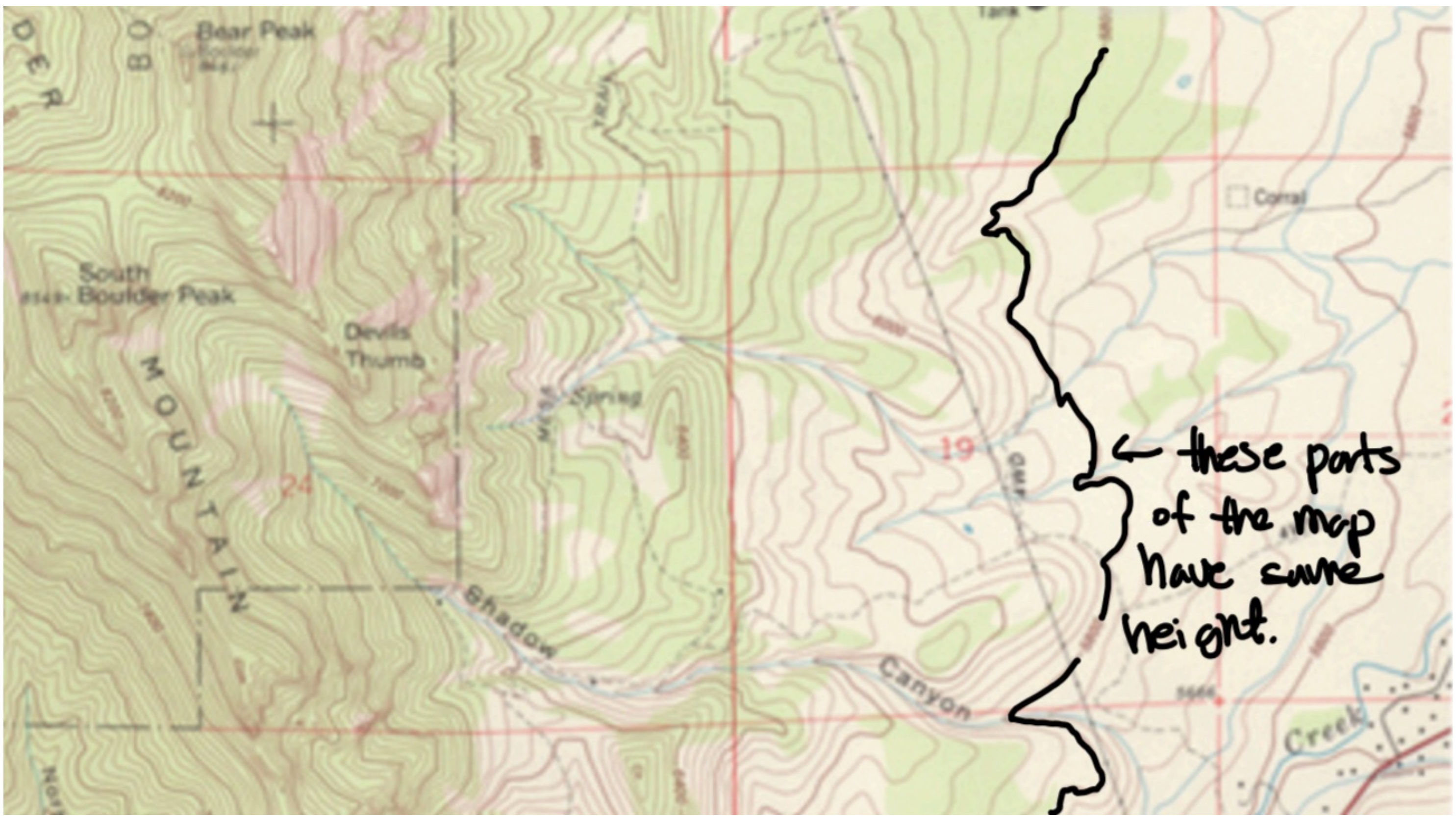


Level Curves

For a function of two variables $f(x, y)$ a level curve is the set of points (x, y) in the xy-plane where $f(x, y)$ is equal to a constant z_0.



all the point on the green curve has same height (z value) on the graph.



Bear Peak
South
Boulder Peak
Devils
Thumb
MOUNTAIN
Spring
Corral
19
24
Shadow
Canyon
Creek
← these parts
of the map
have same
height.

E.g. Sketch graph.

$p(x,y) = \sqrt{x^2+y^2}.$

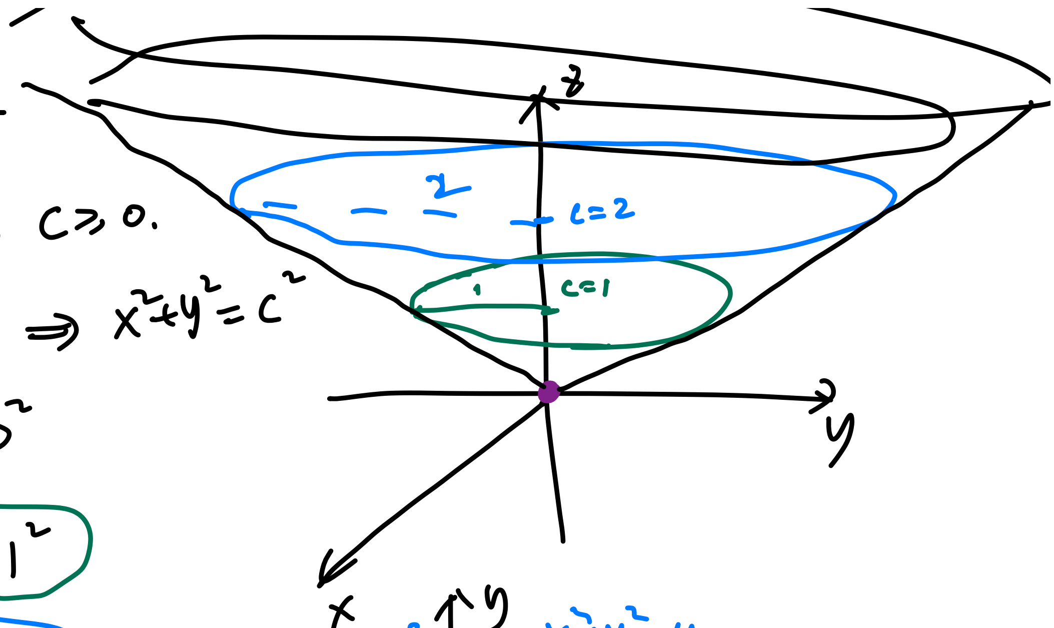
Level curves: $c \geq 0.$

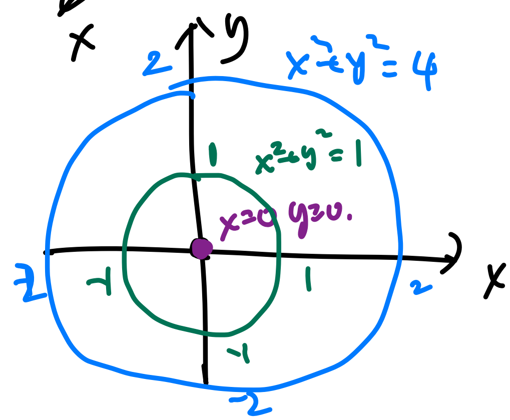
$\sqrt{x^2+y^2} = c \Rightarrow x^2+y^2 = c^2$

$c=0: \quad x^2+y^2 = 0^2$

$c=1: \quad x^2+y^2 = 1^2$

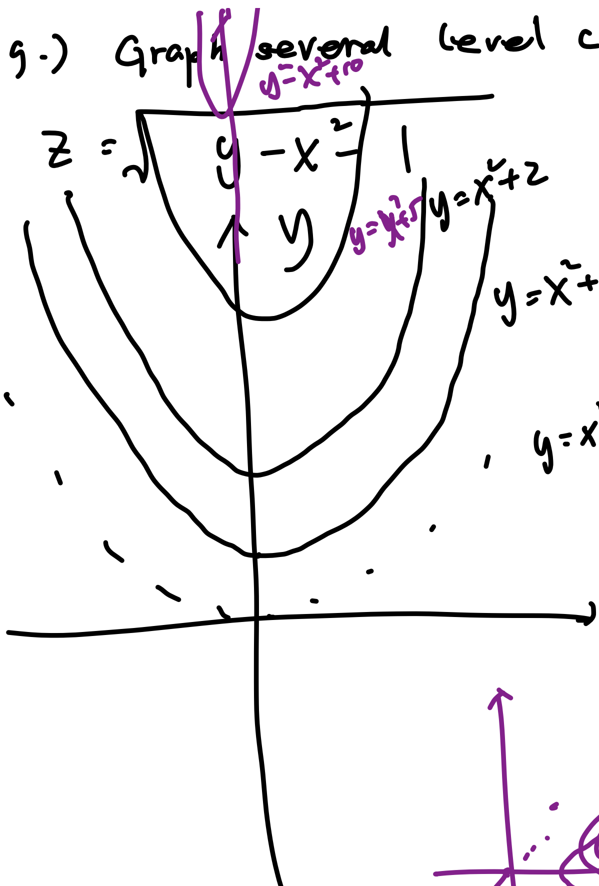
$c=2: \quad x^2+y^2 = 4.$





E.g.) Graph several level curves

$z = \sqrt{y - x^2 - 1}$



Level curves $\sqrt{y - x^2 - 1} = c \quad (c \geq 0)$.

$$y - x^2 - 1 = c^2$$

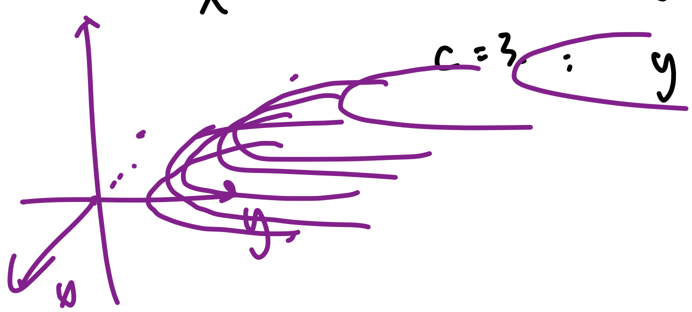
$$y = x^2 + 1 + c^2$$

$c = 0$: $\underline{\underline{y = x^2 + 1}}$

$c = 1$: $y = x^2 + 1 + 1 = x^2 + 2$.

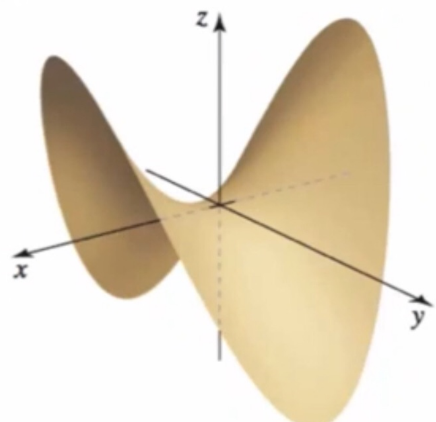
$c = 2$: $y = x^2 + 1 + 4 = x^2 + 5$

$c = 3$: $y = x^2 + 1 + 9 = x^2 + 10$

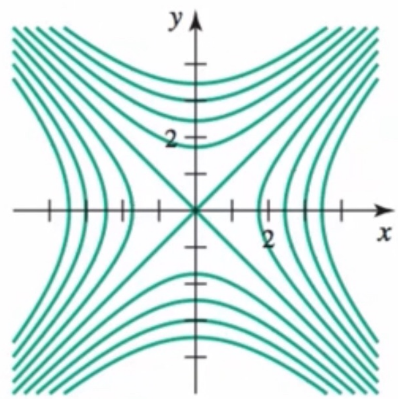


Level Surface

Need 3 dimensions to plot the graph of $z = f(x, y)$



Need 2 dimensions to plot level curves of $z = f(x, y)$



Need 4 dimensions to plot the graph of $w = f(x, y, z)$

Need 3 dimensions to plot level **surfaces** of $w = f(x, y, z)$

Functions of More Than Two Variables

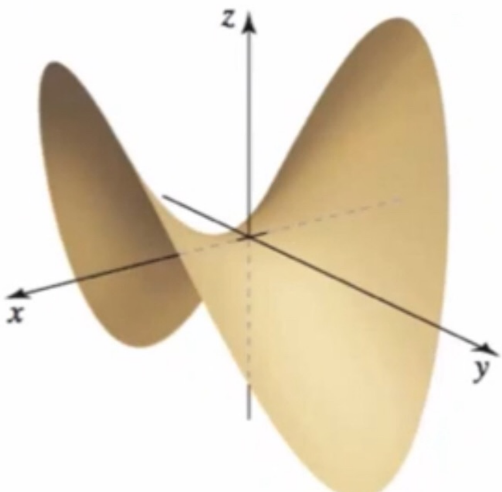
- Three variables: $w = f(x, y, z)$
- n variables: $x_{n+1} = f(x_1, x_2, \dots, x_n)$

DEFINITION Function, Domain, and Range with n Independent Variables

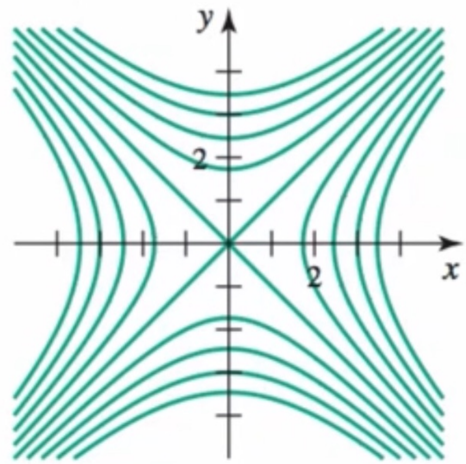
The **function** $x_{n+1} = f(x_1, x_2, \dots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \dots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \dots, x_n)$ vary over the domain.

Level Surface

Need 3 dimensions to plot the graph of
$z = f(x, y)$



Need 2 dimensions to plot level curves of
$z = f(x, y)$



Need 4 dimensions to plot the graph of
$w = f(x, y, z)$

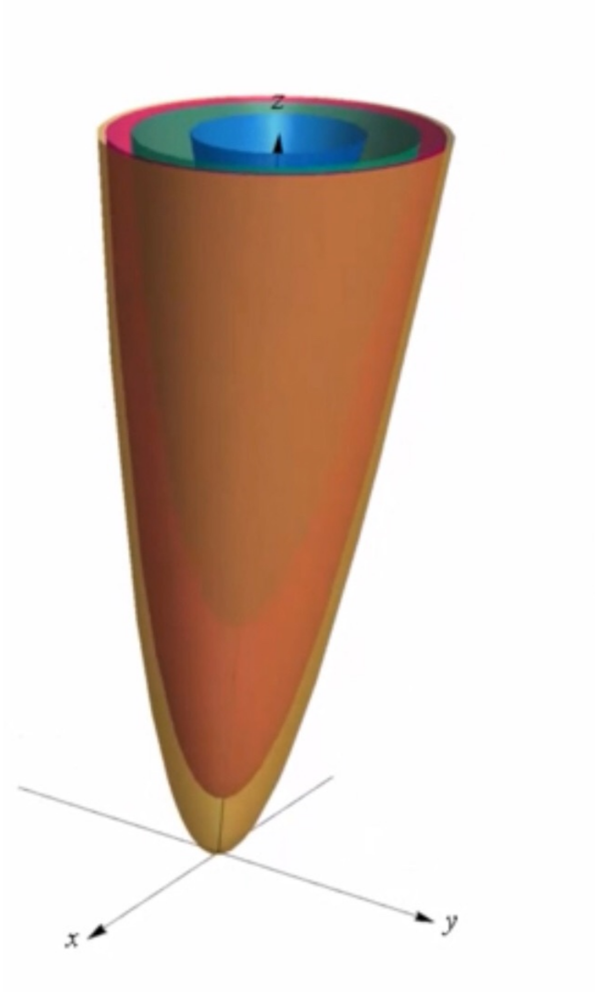
$x^2 + y^2 + z = 1$

Need 3 dimensions to plot level **surfaces** of
$w = f(x, y, z)$

Level Surface

For a function of three variables $f(x, y, z)$ a level surface is the set of points (x, y, z) in xyz-space where $f(x, y, z)$ is equal to a constant w_0.

e.g



Level surface.

$w = f(x,y,z), = \sqrt{z - x^2 - 2y^2}.$ $=3$

$=9$

level surface

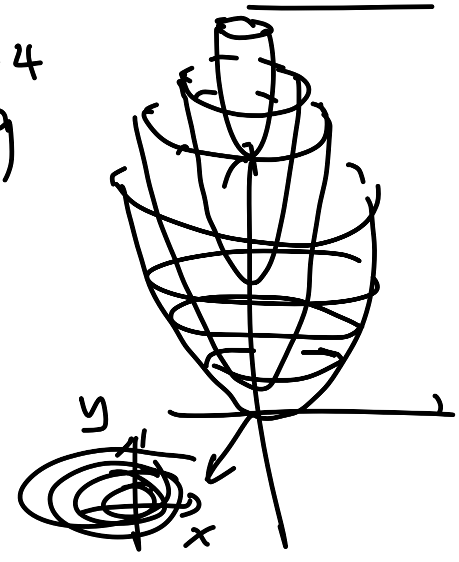
$w=0$ $\quad \sqrt{z - x^2 - 2y^2} = 0 \Rightarrow z - x^2 - 2y^2 = 0$

$\Rightarrow \underline{z = x^2 + 2y^2}.$

$\underline{w=1}$ $\quad z = x^2 + 2y^2 + 1$

$\underline{w=2}$ $\quad z = x^2 + 2y^2 + 4$

$w=3.$ $\quad z = x^2 + 2y^2 + 9$



Eg. Find the domain

$$f(x,y,z)=\frac{10}{x^2-(y+z)x+yz}\neq 0.$$

$$=\frac{10}{(x-y)(x-z)}.$$

$x^2\boxed{-(y+z)x}+yz$

$x \quad -y \quad (x-y)$

$x \quad -z \quad (x-z)$

$-zx-yx$

$=-(y+z)x$

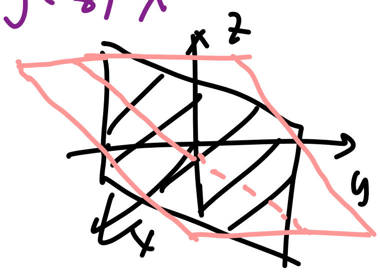
Denominator $\neq 0$

$(x-y)(x-z)\neq 0$

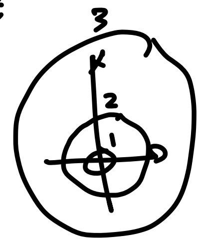
$x-y\neq 0$ and $x-z\neq 0$

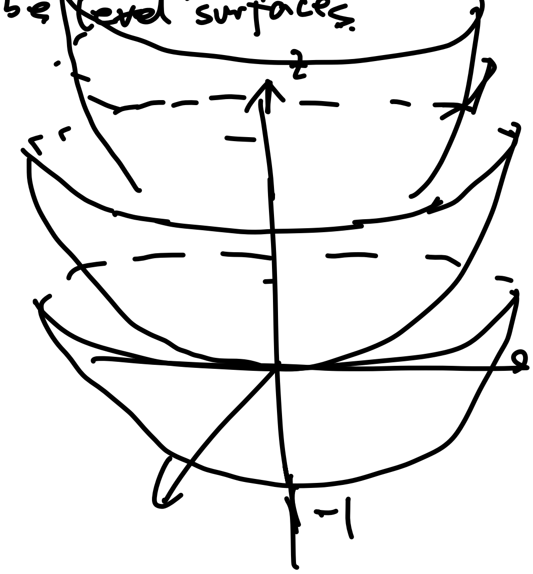
$x\neq y$ and $x\neq z$.

$$\boxed{D=\{(x,y,z): x\neq y,\ x\neq z\}}$$



E.g. $w = f(x, y, z) = x^2 + y^2 - z$ Describe level surfaces.





$\underline{w=0}$: $0 = x^2 + y^2 - z$

$z = x^2 + y^2$.

$\underline{w=1}$: $1 = x^2 + y^2 - z$

$z = x^2 + y^2 - 1$

$w = -1$: $-1 = x^2 + y^2 - z$

$z = x^2 + y^2 + 1$

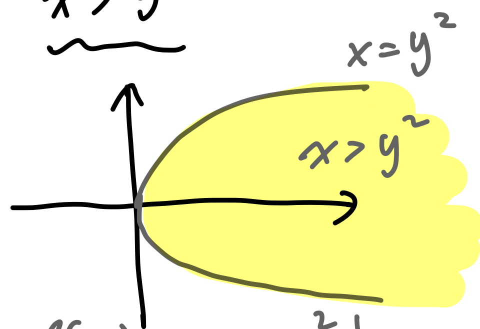
level surfaces : Paraboloid of the form

$z = x^2 + y^2 + c$.

$z = \ln(x - y^2)$

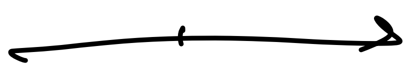
Domain : $x - y^2 > 0$

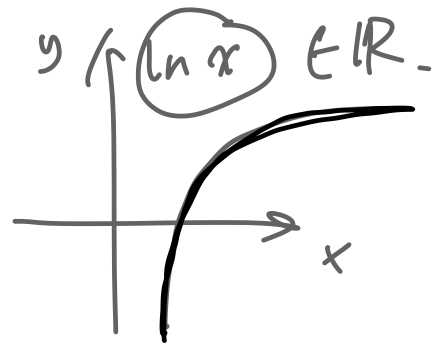
$x > y^2$



$D = \{(x, y) : x > y^2\}$.

Range : $z \in \mathbb{R}$

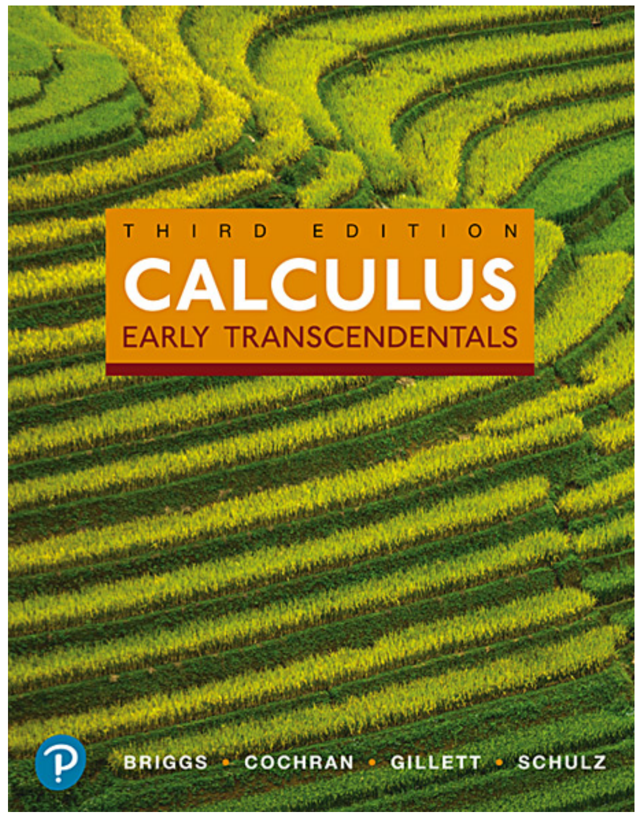




$x > 0$.

15.2

Limits and Continuity



$f(x)$.

$$\lim_{x \to a} f(x) = L.$$

$\forall \varepsilon > 0$ $\quad\exists$ there exists a corresponding number $\delta > 0$

for any

such that. $\quad |f(x) - L| < \varepsilon. \qquad 0 < |x - a| < \delta$

s.t.

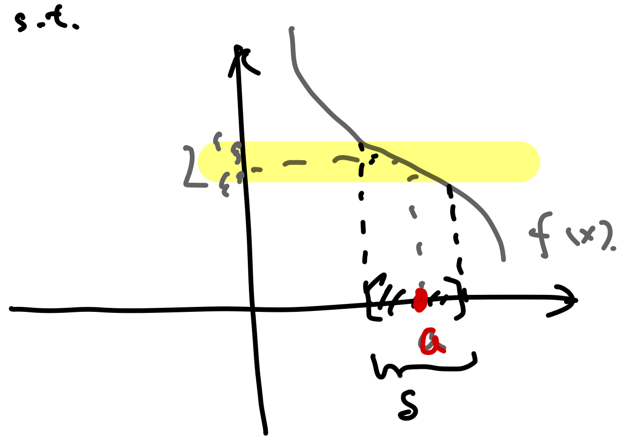
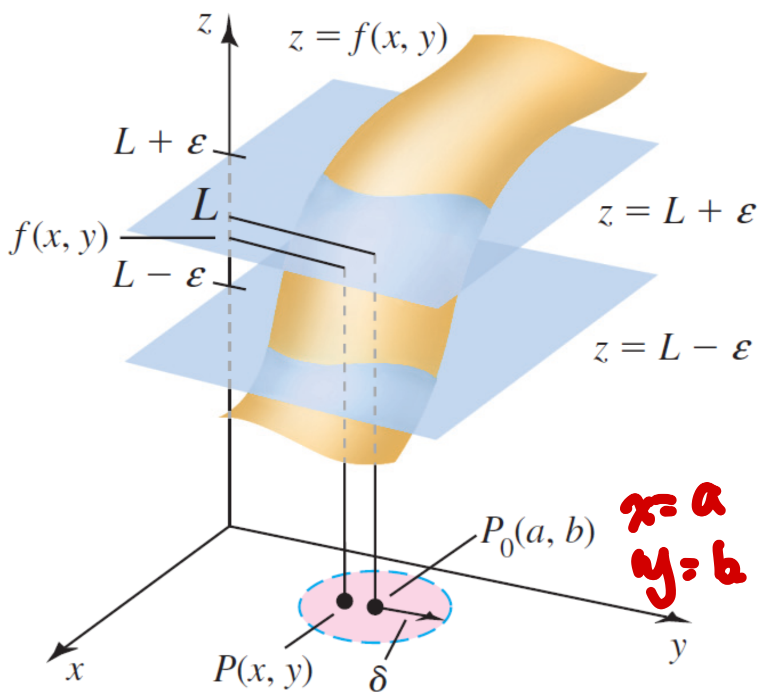


Figure 15.20



$f(x, y)$ is between $L - \varepsilon$ and $L + \varepsilon$ whenever $P(x, y)$ is within δ of P_0.

DEFINITION Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L,$$

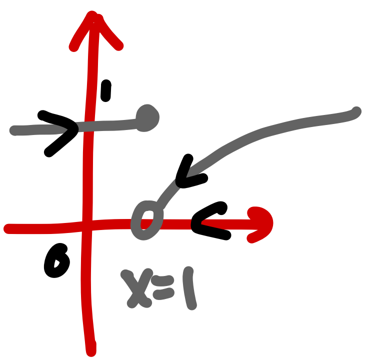
if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

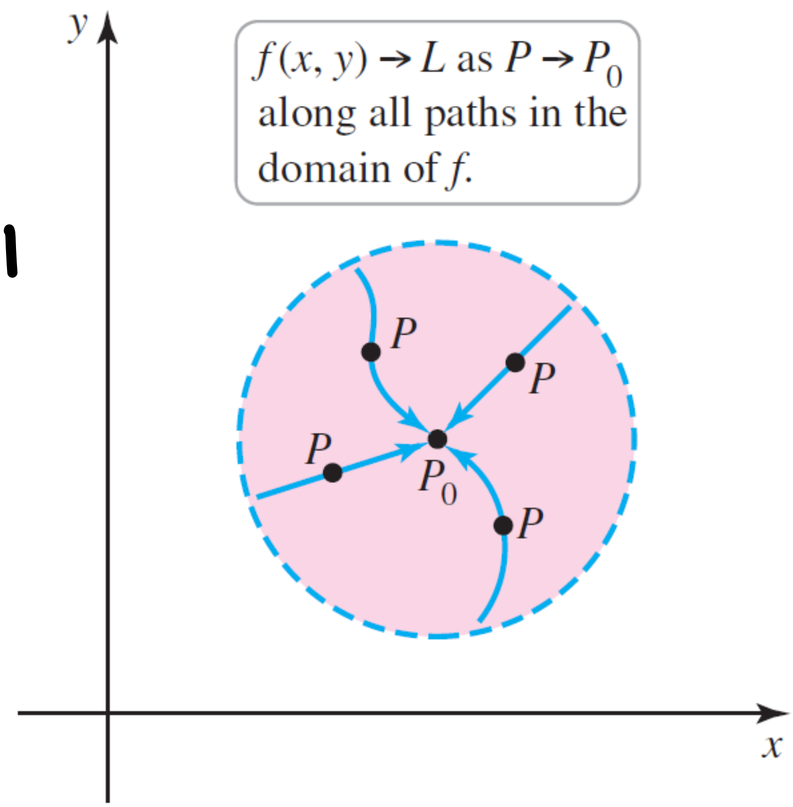
$$0 < |PP_0| = \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

f(x)



DNE at x=1

Figure 15.21



THEOREM 15.1 Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant function $f(x, y) = c$: $\displaystyle\lim_{(x, y)\to(a, b)} c = c$
2. Linear function $f(x, y) = x$: $\displaystyle\lim_{(x, y)\to(a, b)} x = a$
3. Linear function $f(x, y) = y$: $\displaystyle\lim_{(x, y)\to(a, b)} y = b$

THEOREM 15.2 Limit Laws for Functions of Two Variables

Let L and M be real numbers and suppose $\lim_{(x, y) \to (a, b)} f(x, y) = L$ and $\lim_{(x, y) \to (a, b)} g(x, y) = M$. Assume c is a constant, and $n > 0$ is an integer.

1. **Sum** $\lim_{(x, y) \to (a, b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim_{(x, y) \to (a, b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim_{(x, y) \to (a, b)} cf(x, y) = cL$
4. **Product** $\lim_{(x, y) \to (a, b)} f(x, y)g(x, y) = LM$
5. **Quotient** $\lim_{(x, y) \to (a, b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x, y) \to (a, b)} (f(x, y))^n = L^n$
7. **Root** $\lim_{(x, y) \to (a, b)} (f(x, y))^{1/n} = L^{1/n}$, where we assume $L > 0$ if n is even.

Example - Limit of a function of two variables

Evaluate: $\lim\limits_{(x,y)\to(2,0)} \dfrac{x^2 - 3xy^2}{x + y} = \dfrac{2^2 - 3\cdot 2\cdot 0^2}{2+0} = \dfrac{4}{2}$

$x = 2$
$y = 0$

$= 2$

$\lim\limits_{(x,y)\to(2,4)} 452 = 452.$

$x = 2$
$y = 4$

$\dfrac{x^2 - y^2}{x - y} = \dfrac{(x+y)\cancel{(x-y)}}{\cancel{x-y}} = x+y$

$\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2 - y^2}{x - y} = \dfrac{1^2 - 1^2}{1-1} = \boxed{\dfrac{0}{0}}$

$\lim\limits_{(x,y)\to(1,1)} x+y = 1+1 = \boxed{2}$

DEFINITION Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R (Figure 15.22).

A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Figure 15.22

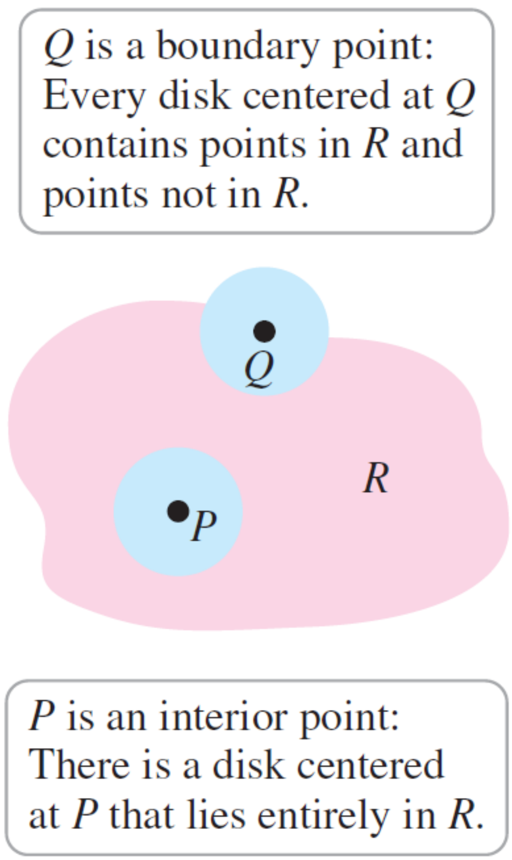
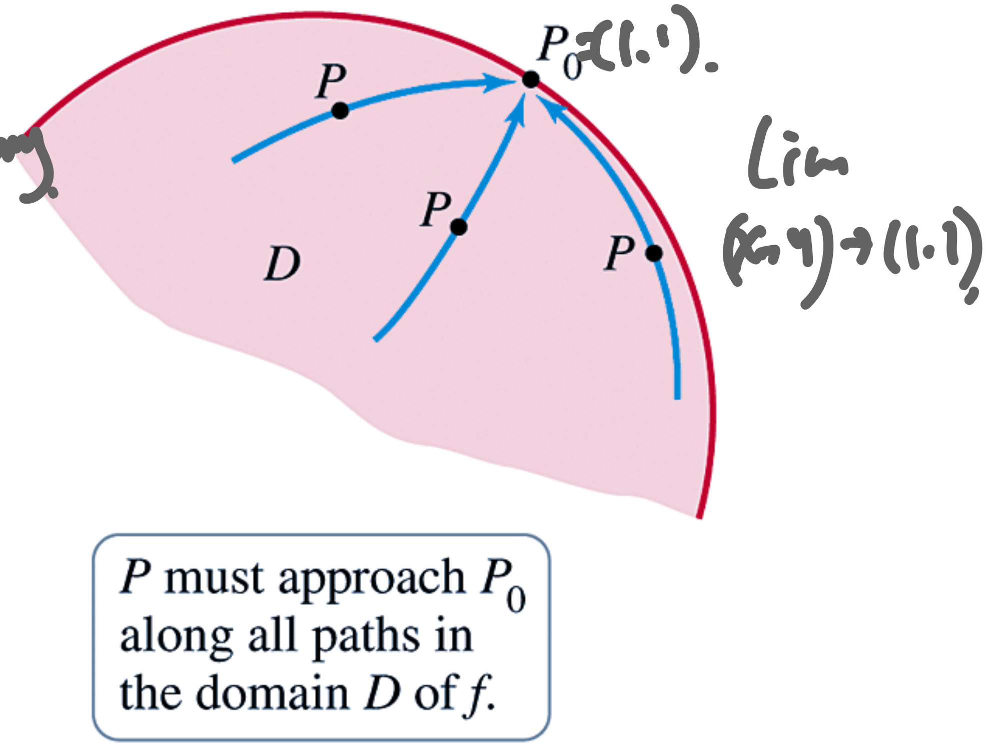

Q is a boundary point:
Every disk centered at Q contains points in R and points not in R.
Q
R
P
P is an interior point:
There is a disk centered at P that lies entirely in R.

Figure 15.23

$$\frac{x^2-y^2}{x-y}.$$

x=y boundary

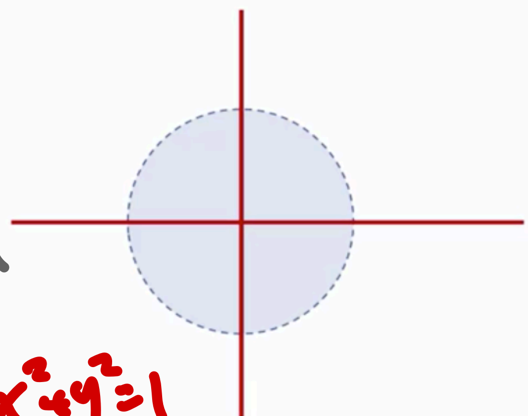


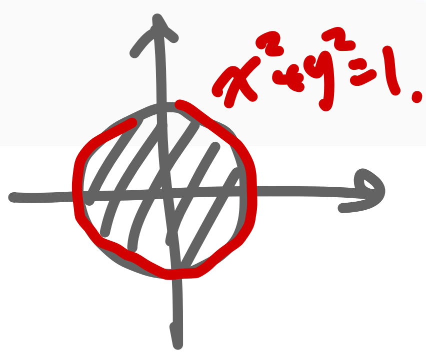
DEFINITION **Open and Closed Sets**

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

$\{(x, y) : x^2 + y^2 < 1\}$ ← open.

$\{(x, y) : x^2 + y^2 \leq 1\}$ ← closed





Example - Evaluate a limit at boundary point of domain

$$\lim_{(x,y)\to(-1,1)} \frac{2x^2 - xy - 3y^2}{x + y} = \frac{2\cdot(-1)^2 - (-1)\cdot 1 - 3\cdot 1^2}{-1+1} = \frac{2+1-3}{0} = \frac{0}{0}.$$

$$= \lim_{(x,y)\to(-1,1)} \frac{\cancel{(x+y)}(2x-3y)}{\cancel{(x+y)}}$$

$$= 2(-1) - 3\cdot 1 = \boxed{-5}$$

$$2x^2 - xy - 3y^2 = (x+y)(2x-3y).$$

$$\begin{array}{ccc} x & & y \\ & \times & \\ 2x & & -3y \end{array}$$

$$-3xy + 2xy = -xy$$

Example - Limit of a function of three variables

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2+xy-xz-yz}{x-z} \Rightarrow \frac{1+1-1-1}{1-1} = \frac{0}{0}.$$

$x=1$
$y=1$
$z=1$

$$x^2+xy-xz-yz = x(x-z)+y(x-z)$$
$$=(x+y)(x-z).$$

$$\lim \frac{(x+y)\cancel{(x-z)}}{\cancel{(x-z)}} = \lim_{(x,y,z)\to(1,1,1)} x+y = 1+1 = \boxed{2}.$$

Example - Nonexistence of limits, Two-Path Test

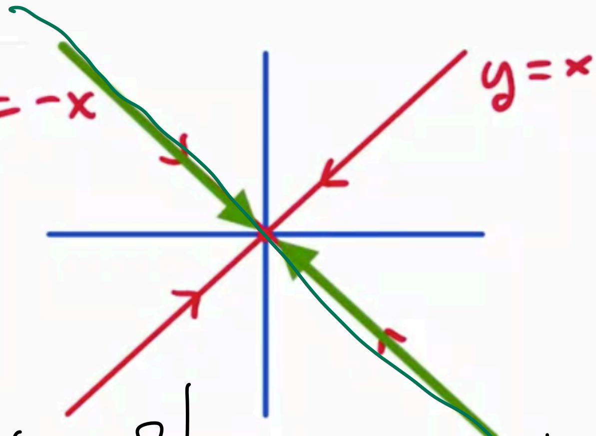
$$\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{xy^2} = \frac{0^3 + 0^3}{0 \cdot 0^2} = \frac{0}{0}.$$

Example - Nonexistence of limits, Two-Path Test

$$\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{xy^2}$$

Example - Nonexistence of limits, Two-Path Test

$$\lim_{(x,y)\to(0,0)} \frac{y^3+x^3}{xy^2} \qquad \boxed{\text{DNE.}}$$



① $y=x$

what value $\dfrac{y^3+x^3}{xy^2}$ approach

as $(x,y)\to(0,0)$ along $\boxed{y=x}$ line?

$$\lim_{\substack{x\to y,(0,0)\\ y=x}} \frac{y^3+x^3}{xy^2} \underset{y=x}{=} \lim_{x\to 0}\frac{x^3+\cancel{y}^3}{x\,x^2} = \lim_{x\to 0}\frac{2x^3}{x^3} = \lim 2 = \boxed{2}$$

as $(x,y)\to(0,0)$ along $y=$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} \frac{y^3+x^3}{xy^2} \underset{y=-x}{=} \lim_{y\to 0}\frac{(-x)^3+x^3}{x(-x)^2} = \lim_{y\to 0}\frac{0}{x^3} \overset{\lim}{=} 0 = \boxed{0}$$

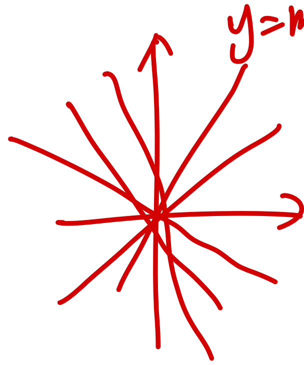
② $y=-x$ what value $\dfrac{y^3+\cancel{y}^3}{\cancel{x}y^2}$ approach

PROCEDURE **Two-Path Test for Nonexistence of Limits**

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

$f(x,y) = \frac{y+x}{y-2x}$.

$$\lim_{(x,y)\to(0,0)} \frac{y+x}{y-2x} =$$



Two-path test.

Consider $y = mx$ path. ($m \neq 2$)

$x, y \to (0,0)$ along $y = mx$.

$y = 2x$, is not in domain

$m \neq 2$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{y+x}{y-2x} = \lim_{x\to 0} \frac{mx+x}{mx-2x} = \lim_{x\to 0} \frac{(m+1)x}{(m-2)x}$$

$$= \lim_{x\to 0} \frac{m+1}{m-2}$$

$$= \frac{m+1}{m-2}$$

when $m=1$ $\lim = \frac{1+1}{1-2} = \boxed{-2}$

$\neq$

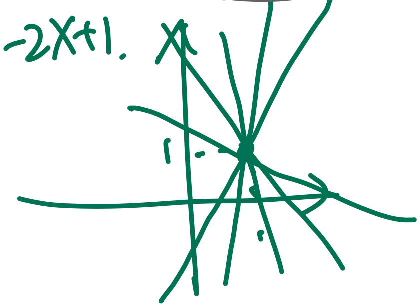
when $m=0$ $\lim = \frac{0+1}{0-2} = \boxed{-\frac{1}{2}}$

DNE.

$y - 2x + 1 = 0$.

$$\lim_{(x,y)\to(1,1)} \frac{y+x+1}{y-2x+1}.$$

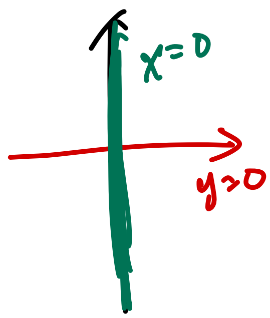
$y = 2x - 1$



$$f(x,y) = \frac{y^4 - 2x^2}{y^4 + x^2}.$$

$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2}$$

Two-path Test



• what value $f(x,y)$ approach as $(x,y)\to(0,0)$ along x-axis? $\boxed{y=0}$ path

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{y^4-2x^2}{y^4+x^2} = \lim_{x\to0} \frac{0^4-2x^2}{0^4+x^2}.$$

$$= \lim_{x\to0} \frac{-2\cancel{x^2}}{\cancel{x^2}} = \lim_{x\to0} -2 = \boxed{-2}$$

• what value $f(x,y)$ approach as $(x,y)\to(0,0)$ along y-axis? $\boxed{x=0}$ path.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{y^4-2x^2}{y^4+x^2}$$

$$= \lim_{y\to0} \frac{y^4-2\cdot0^2}{y^4+0^2}$$

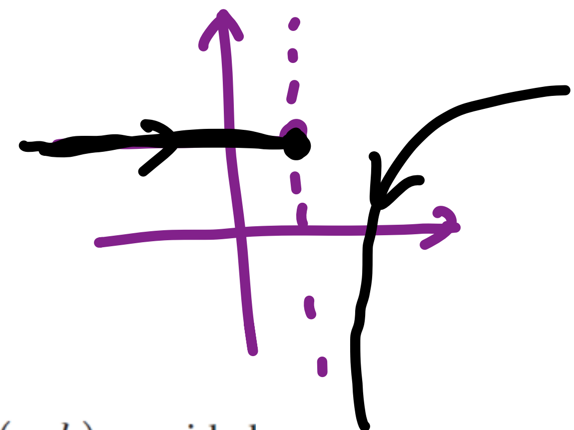
$$= \lim_{y\to0} \frac{\cancel{y^4}}{\cancel{y^4}}$$

$$= \lim_{y\to0} 1 = \boxed{1}$$

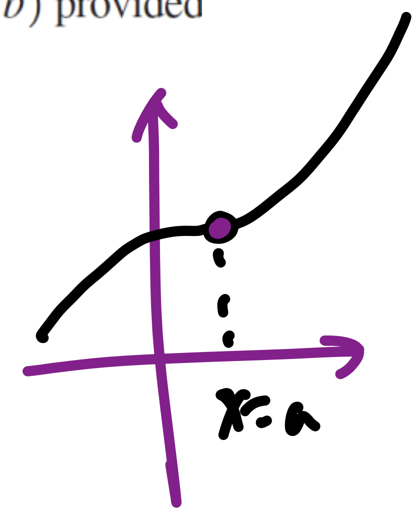
$\neq$ DNE.

DEFINITION Continuity

The function f is continuous at the point (a, b) provided



1. f is defined at (a, b).
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

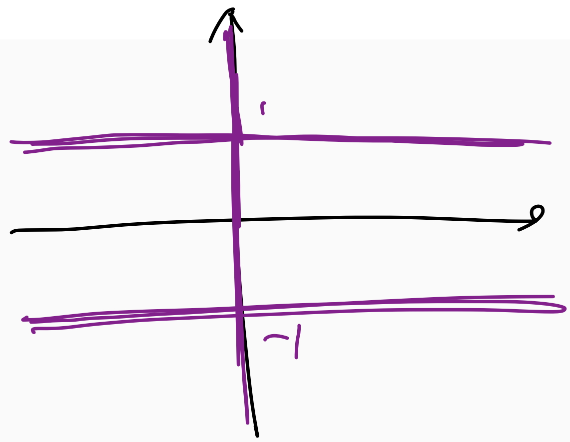


Example - Determine points of continuity

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{x(y^2-1)} \\ 20 \end{cases}$$

$x \neq 0 \quad y \neq 0$

$x = 0 \quad y = 0.$



Domain: $x(y^2-1) \neq 0.$

$x \neq 0 \quad y^2 - 1 \neq 0$

$x \neq 0$ and $y \neq -1$ and $y \neq 1$

$\lim \frac{x^2+y^2}{x(y^2-1)}$ exists.

points $D = \{(x,y) : x \neq 0,\ y \neq -1,\ y \neq 1\}$, $f(x,y)$ Continuous.

THEOREM 15.3 Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

THEOREM 15.3 Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

e.g.

At what points of $\mathbb{R}^2$ is the following function continuous?

① $f(x,y) = 2x^2 - 3xy + 4y$.

- All points of $\mathbb{R}^2$.

② $p(x,y) = \dfrac{16x^4y^2}{x^4+y^2}$ $\qquad x^4+y^2 \neq 0 \quad (x,y) \neq (0,0)$

- The function is continuous at all points of $\mathbb{R}^2$ except $(0,0)$.

③ $g(x,y) = \ln(y-2x)$

- All points of $\mathbb{R}^2$ satisfying the inequality $y-2x > 0$.

15.3 Partial Derivatives

$f(x, y)$

Partial derivati of f with r to x at (a, b) is

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

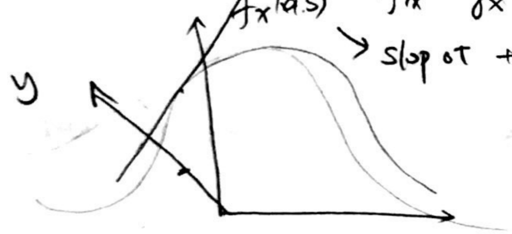
wrt y

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

provide these limit exist

$f_x = \frac{\partial f}{\partial x}$ $\quad$ $f_y = \frac{\partial f}{\partial y}$

$f_x(a,b)$ → slop of tagent line.



e.g. $\quad f(x,y) = x + y^2 + 4$

$f_x(x,y) = 1$ $\qquad$ $f_y(x,y) = 2y$

e.g. $\quad$ Partial deratives via differentia rules

$f(x,y) = (3xy + 4y^2 + 1)^5$

$$f_x(x,y) = 5(3xy + 4y^2 + 1)^4 \underbrace{\frac{\partial}{\partial x}(3xy + 4y^2 + 1)}_{3y+0+0}$$

$$= 5(3xy + 4y^2 + 1)^4 \cdot 3y$$

$$f_y(x,y) = 5(3xy + 4y^2 + 1)^4 \underbrace{\frac{\partial}{\partial y}(3xy + 4y^2 + 1)}_{3x+8y}$$

$$= 5(3xy + 4y + 1)(3x + 8y)$$

Higher order partials

$(f_x)_x = f_{xx}$ $\qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$(f_x)_y = f_{xy}$ $\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ $\qquad$ mix partial

$(f_y)_x = f_{yx}$ $\qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

$(f_y)_y = f_{yy}$ $\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial y}$

---

e.g. $f(x,y) = \cos(xy)$

$f_x(x,y) = -y\sin(xy)$ $\xrightarrow{\frac{\partial}{\partial x}}$ $f_{xx}(x,y) = -y^2\cos(x,y)$

$\xrightarrow{\frac{\partial}{\partial y}}$ $f_{xy}(x,y) = \boxed{-\sin(xy) - xy\cos(xy)}$

$f_y(x,y) = -x\sin(xy)$ $\xrightarrow{\frac{\partial}{\partial x}}$ $f_{yx}(x,y) = \boxed{-\sin(xy) - xy\cos(x,y)}$

$\xrightarrow{\frac{\partial}{\partial y}}$ $f_{yy}(x,y) = -x^2\cos(x,y)$

Thm
If f define on $D \subset \mathbb{R}^2$, f_{xy} and f_{yx} are continuous on D

Then $f_{xy} = f_{yx}$ on D

$G(r,s,t) = \sqrt{r s^3 t^5} = \sqrt{r}\cdot\sqrt{s^3 t^5} = r^{\frac{1}{2}}\sqrt{s^3 t^5} = s^{3/2}\sqrt{r t^5}$
$= t^{5/2}\sqrt{r s^3}$

$G_r = \frac{1}{2} r^{-\frac{1}{2}}\sqrt{s^3 t^5} = \frac{1}{2\sqrt{r}}\sqrt{s^3 t^5} = \frac{1}{2}\sqrt{\frac{s^3 t^5}{r}}$

$G_s = \frac{3}{2} s^{1/2}\sqrt{r t^5} = \frac{3}{2}\sqrt{r s t^5}$

$G_t = \frac{5}{2} t^{3/2}\sqrt{r s^3} = \frac{5}{2}\sqrt{t^3}\sqrt{r s^3} = \frac{5}{2}\sqrt{r s^3 t^3}$

$z=f(x,y)$ Differential at (a,b)

Thm – Suppose f has partial derivative f_x and f_y defined on open set D $(a,b)\in D$

with f_x and f_y continuous at (a,b)

Then f is differentiable at (a,b)

Thm $D \Rightarrow C$

If f is differential at (a,b) The f is continuous at (a,b)

e.g $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) \text{ DNE}}$ by two path test

Along $y=x$ $\lim_{x\to 0} f(x,x) = 0$

Along $x=y^2$. $\lim_{y\to 0} f(y^2,y) = 1$

$\Rightarrow$ not continuous $\Rightarrow$ not differential

But Both f_x + f_y exist at $(0,0)$

$f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{0}{h^2}-0}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$

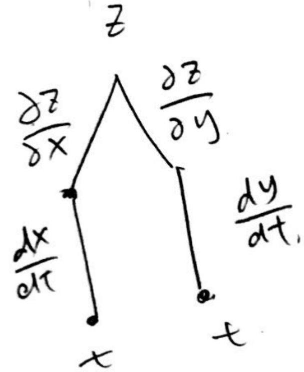
$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{0}{h^4}-0}{h} = \lim_{h\to 0} \dfrac{0}{h^5} = 0.$

15.4

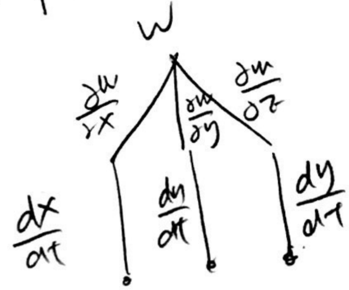
Chain Rule (One Indep Var.

$z = f(x, y) \quad x = g(t) \quad y = h(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



$w = f(x, y, z)$; x, y, z depent on t



$$\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

e.g.e $\frac{dz}{dt}$ when $z = x\sin y$, $x = t^2$ $y = 4t^3$.

$\frac{\partial z}{\partial x} = \sin y \qquad \frac{dx}{dt} = 2t$

$\frac{\partial z}{\partial y} = x\cos y \qquad \frac{dy}{dt} = 12t^2$.

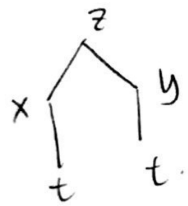
$$\frac{dz}{dt} = \sin y \cdot 2t + x\cos y \cdot (12t^2)$$

$$= \sin(4t^3)2t + t^2\cos(4t^3) \cdot 12t^2$$

Chain Rule 15.4

Example Chain Rule (one independent variable)

1. Find: $\frac{dz}{dt}$ where $z = x\sin y$, $x = t^2$, $y = 4t^3$



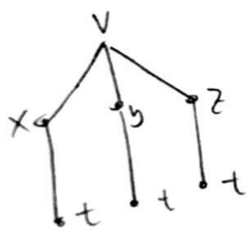
$\frac{\partial z}{\partial x} = \sin y \qquad \frac{\partial z}{\partial y} = x\cos y$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 12t$

$$\frac{dz}{dt} = (\sin y)(2t) + (x\cos y)(12t)$$

$$= \sin(4t^3)\,2t + t^2\cos(4t^2)\cdot 12t^2$$

2. Find $\frac{dV}{dt}$ where $V = xyz$, $x = e^t$, $y = 2t+3$, $z = \sin t$



$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial V}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial V}{\partial z}\cdot\frac{dz}{dt}.$$

$\frac{\partial V}{\partial x} = yz \qquad \frac{\partial V}{\partial y} = xz \qquad \frac{\partial V}{\partial z} = xy.$

$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = 2 \qquad \frac{dz}{dt} = \cos t.$

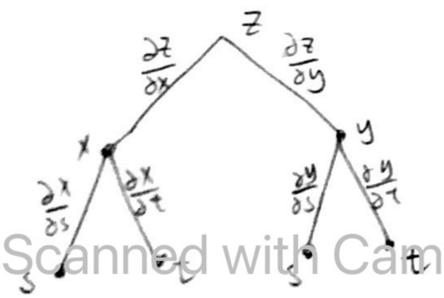
$$\frac{dV}{dt} = yze^t + 2xz + xy\cos t.$$

$$= (2t+3)\sin t\, e^t + 2\sin t\, e^t + e^t(2t+3)\cos t$$

$$= e^t((2t+5)\sin t + (2t+3)\cos t)e^t.$$

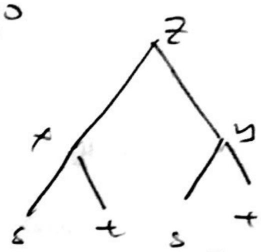
Chain Rule – Two independent Variables $\quad z = f(x,y),\ x = g(s,t),\ y = h(s,t)$

<u>Thm</u> $\quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$



Example: Chain Rule (two independent variables)

Find: z_s and z_t, where $z = \sin xy$, $x = s^2 t$, $y = (s+t)^{10}$



$$\frac{\partial z}{\partial x} = y\cos(xy) \qquad \frac{\partial z}{\partial y} = x\cos(xy)$$

$$\frac{\partial z}{\partial s} = 2st \qquad \frac{\partial x}{\partial t} = s^2 \qquad \frac{\partial y}{\partial s} = 10(s+t)^9 \qquad \frac{\partial y}{\partial t} = 10(s+t)^9$$

$$z_s = \left(y\cos(x.y)\right)(2st) + \left(x\cos(xy)\right)\left(10(s+t)^9\right)$$

$$= \cos(xy)\left[2sty + 10x(s+t)^9\right]$$

$$= \cos\left[s^2t(s+t)^{10}\right]\left[2st(s+t)^{10} + 10s^2 + (s+t)^9\right]$$

$$z_t = (y\cos(xy))\cdot(s^2) + x\cos(xy)\cdot\left(10(s+t)^9\right).$$

Implicit Differentiation

Thm Let F be differentiale on D

$F(x,y) = 0$ $\quad$ y is a differentiable funct. of x

$F_y \neq 0$

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$

$$\therefore \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$F_x - F_y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example : implicit differentiel

Find $\frac{dy}{dx}$ if $\underbrace{ye^{xy}-2}_{F(x,y)}=0$

---

sol: $F_x = y^2e^{xy}$

$F_y = e^{xy} + yxe^{xy}$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2e^{xy}}{e^{xy}+y\,x\,e^{xy}} = -\frac{y^2}{1+yx}$$

Example Walking on a surface

surface $z = x^2+4y^2+1$

oriented curve C in xy-plane

$x=\cos t \quad y=\sin t, \quad 0\le t\le 2\pi$

a) Find $z'(t)$

b) Imagine walking on surface directly above C in positive oritation

Find t which you are walking uphill.

---

a) $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x(-\sin t) + 8y(\cos t)$$

$$= -2\cos t\sin t - 8\sin t\cos t$$

$$= -10\cos t\sin t$$

b) $\frac{dz}{dt} > 0$

$-10\cos t\sin t > 0$ if $\frac{\pi}{2} < t < \pi$

$\frac{3\pi}{2} < t < 2\pi$